Fuzzy analysis as alternative to stochastic methods – theoretical aspects

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Abstract:

A realistic and reliable numerical simulation demands suitable computational models and applicable data models for the structural design parameters. Structural design parameters are in general non-deterministic, i.e. uncertain. The choice of an appropriate uncertainty model for describing selected structural design parameters depends on the characteristic of the available information. Besides the most often used probabilistic models and the related stochastic analysis techniques newer uncertainty models offer the chance taking account of non-stochastic uncertainty that frequently appears in engineering problems.

The uncertainty model fuzziness and the algorithm of the fuzzy structural analysis is presented in this paper. The uncertainty quantification of real-world data for the uncertainty models fuzziness and randomness is discussed by the way of examples. The differences and advantages of uncertainty models randomness and fuzziness and its simulation techniques are addressed.

Keywords:

uncertainty, uncertainty quantification, fuzziness, fuzzy structural analysis, randomness

1 Introduction

Structural engineering mainly focuses on computing structural responses, assessing structural safety, and determining parameters for structural design that meets all relevant requirements. For this purpose, the structural engineer has to apply appropriate structural models, suitably-matched computational models and reliable structural parameters close to reality. Structural models and structural parameters have to be established on the basis of plans, drawings, measurements, observations, experiences, expert knowledge, codes and standards. As a rule certain information regarding structural models and precise values of structural parameters do not exist. Computational models must be capable of numerically simulating the system behavior of the chosen structural model. Mathematically exact solutions, however, are only available in exceptional cases. In general, weak solutions and approximations are used, internal parameters, e.g. in material laws, have to be defined and numerical solution techniques including lower bounds for numerical accuracy are applied. These facts show that structural engineering is significantly characterized by uncertainty. In order to perform realistic structural analysis and safety assessment this uncertainty must be appropriately taken into consideration.

Different methods are available for mathematically describing and quantifying uncertainty. Some of these basic concepts are e.g. probability theory [20], including subjective probability approach [34] and BAYES methods [30], interval mathematics [1], convex modeling [6], theory of rough sets [25], fuzzy set theory [2], theory of fuzzy random variables [17] and chaos theory [16]. In the scientific literature the new uncertainty models are not only controversially discussed [10] but also increasingly implemented for the solution of practice-relevant problems [27, 9, 5, 8, 15, 24, 31, 26]. These different developments of uncertainty models do not directly contradict each other but rather constitute an entirety.

The choice of an appropriate uncertainty model for solving a particular problem depends on the characteristic of the uncertainty present in the problem description and the boundary conditions. Most often, the well developed probabilistic models are applied to take account of uncertainty. For this purpose, random variables or random processes are generated for describing non-deterministic parameters or parameter fields. This presupposes assured and satisfactory statistical information to estimate the necessary probability distribution functions or special and sophisticated expert knowledge to assume prior distributions for a BAYESian approach. If these prerequisites for dependably applying probabilistic methods are not satisfied, it is advisable to make use of alternative uncertainty models. In this frequent case the engineer has to quantify structural parameters on the basis of only few data, which may additionally be characterized by vagueness, e.g. due to uncertain measurements or changing reproduction conditions. Moreover, some expert knowledge and linguistic assessments are required to be incorporated into the modeling. Hence, the engineer does only have an idea concerning the value range of these parameters and a kind of believe with which some values are more possible to occur than other ones. For modeling such information adequately a non-probabilistic uncertainty model that considers sets of parameter values together with subjective weighting information inside the set is needed. Fuzzy set theory provides the most powerful basis for this purpose. It permits set theoretical modeling of uncertain parameters and a subjective assessment of degrees with which the particular elements belong to the set by means of a membership function. This offers the chance for appropriately taking account of non-stochastic uncertainty, which frequently appears in engineering problems without making any artificial assumptions the validity of which cannot be proven.

This paper mainly focuses on the uncertainty model fuzziness. Numerous information regarding the uncertainty model randomness can be found in literature. The probabilistic analysis methods are also well developed and widespread. For that reason the uncertainty model randomness and the probabilistic analysis methods are not addressed here separately. The application of the uncertainty model fuzziness and the algorithm of fuzzy analysis as described in this paper is demonstrated by the way of an example in the book of proceedings [22] separately.

2 Uncertainty model fuzziness

Uncertain parameters whose uncertainty characteristic has been identified as *fuzziness* are treated on the basis of the fuzzy set theory. In contrast to classical set theory the fuzzy set theory permits a gradual assessment of the membership of elements in relation to set, see Fig. 1. This gradual membership is described by a membership function. A uncertain set is defined as

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in \mathbf{X}\} \tag{1}$$

and also referred as fuzzy set on X. The membership function $\mu_A(x)$ quantifies the grade of membership of the elements x to the fundamental set $\mathbf{X}$. The following holds for the functional values of the membership function $\mu_A(x)$

$$\mu_A(x) \geq 0 \quad \forall \quad x \in \mathbf{X} \tag{2}$$

$$\sup_{x \in X}[\mu_A(x)] = 1 \tag{3}$$

In that case the membership function and the fuzzy set $\tilde{A}$ is referred as normalized and describes the mapping of the elements $x \in \mathbf{X}$ onto the interval $(0,\, 1]$. Subsequently, exclusively normalized membership values are considered.

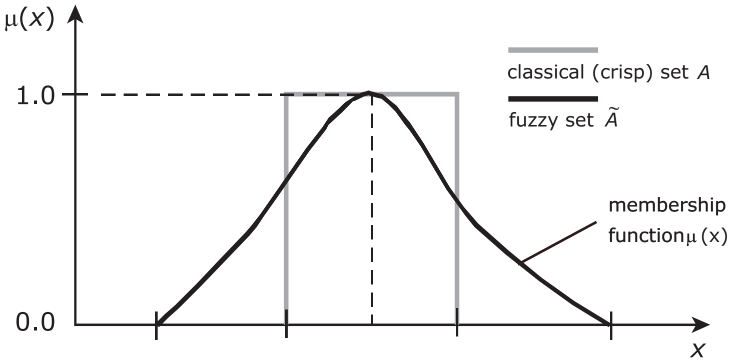


Figure 1: Comparison of a crisp set and a fuzzy set

This formulation of fuzzy set leads to the definition of fuzzy numbers and fuzzy intervals (Fig. 2).

A fuzzy number is a convex, normalized fuzzy set $\tilde{A} \subseteq \mathbb{R}$ whose membership function is at least segmentally continuous and has the functional value $\mu_A(x) = 1$ at precisely one element.

A fuzzy interval is a uncertain set $\tilde{A} \subseteq \mathbb{R}$ with a mean interval whose elements possess the membership function value $\mu_A(x) = 1$. Likewise, the membership function must be convex and normalized and also at least segmentally continuous.

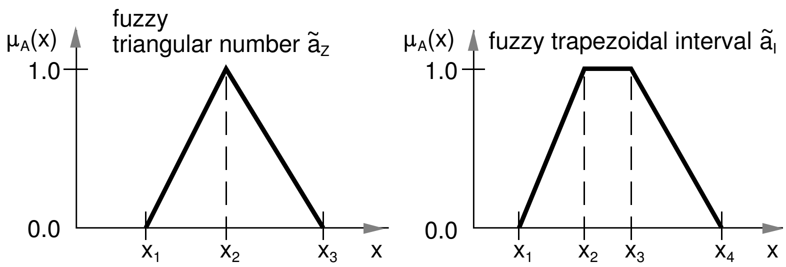


Figure 2: fuzzy number $\tilde{x}_1$ and fuzzy interval $\tilde{x}_2$

Fuzzy functions

A classical function is a single-value mapping of the elements t from the fundamental set $\mathbf{T} \subseteq \mathbb{R}$ onto the elements x of the fundamental set $\mathbf{X} \subseteq \mathbb{R}$. It may be denoted by

$$x(t):\ \mathbf{T} \rightarrow \mathbf{X} \tag{4}$$

A fuzzy function is the mapping of the fuzzy values $\tilde{t}$ of the fundamental set $\mathbf{T}$ on to the fuzzy values $\tilde{x} \in \mathbf{F}(\mathbf{X})$. This definition of $\tilde{x}(\tilde{t})$

$$\tilde{x}(\tilde{t}):\ \mathbf{F}(\mathbf{T}) \tilde{\to} \mathbf{F}(\mathbf{X}) \tag{5}$$

is also referred as *fuzzy function*. A fuzzy function may also interpreted as being a set of fuzzy results or fuzzy functional values $\tilde{x}_t \in \mathbf{F}(\mathbf{X})$ belonging to specified $\tilde{t} \in \mathbf{F}(\mathbf{X})$

$$\tilde{x}(\tilde{t}) = \{\tilde{x}_t = x(\tilde{t})\ \forall\, \tilde{t}\ \mid\ t \in \mathbf{F}(\mathbf{T})\}. \tag{6}$$

Bunch parameter representation. A fuzzy function $\tilde{x}(\underline{t})$ may be formulated depending on fuzzy bunch parameters $\underline{\tilde{s}}$ and the crisp arguments $\underline{t}$

$$\tilde{x}(\underline{t}) = x(\tilde{s}, \underline{t}) = \{\tilde{x}_t = x(\tilde{s}, \underline{t})\ \forall\, \underline{t}\ \mid\ \underline{t} \in \mathbf{T}\} \tag{7}$$

For each crisp bunch parameter vector $\underline{s} \in \underline{\tilde{s}}$ with the membership degree $\mu(s)$ a crisp function $x(\underline{t}) = x(s, \underline{t}) \in \tilde{x}(t)$ with $\mu(x(t)) = \mu(s)$ is obtained. The real functions or elements $\mathrm{x}(\underline{\mathrm{t}})$ of $\tilde{\mathrm{x}}(\underline{\mathrm{t}})$ are defined for all $\underline{\mathrm{t}} \in \underline{\mathbf{T}}$, they are referred to as *trajectories*.

Trajectories of a fuzzy function with discrete arguments are sequences of real numbers, whereas trajectories of a fuzzy function with continuous arguments generally represent continuous functions of $\underline{\mathrm{t}}$. Trajectories of a discrete fuzzy function with continuous arguments are described by step functions.

Example: A one-dimensional fuzzy function

$$\tilde{\mathrm{x}}(\mathrm{t}) = \mathrm{x}(\underline{\tilde{\mathrm{s}}}, \mathrm{t}) = \tilde{\mathrm{a}} \cdot \sin(\tilde{\omega}_0 \mathrm{t} + \tilde{\mathrm{b}}) \tag{8}$$

with the fuzzy bunch parameters $\tilde{\mathrm{a}} =< 0.9, 1.0, 1.1 >$ (fuzzy amplitude), $\tilde{\omega}_0 =< 0.9, 1.0, 1.1 >$ (fuzzy frequency), and $\tilde{\mathrm{b}} =< 0.1, 0.0, 0.1 >$ (fuzzy phase angle) is considered. Within the fuzziness of the fuzzy bunch parameter vector $\underline{\tilde{\mathrm{s}}} = (\tilde{\mathrm{a}}, \tilde{\omega}_0, \tilde{\mathrm{b}})$ different crisp points $\underline{\mathrm{s}} \in \underline{\tilde{\mathrm{s}}}$ with the associated membership values $\mu(\underline{\mathrm{s}})$ may be chosen to generate various trajectories. As $\underline{\tilde{\mathrm{s}}}$ represents a continuous fuzzy set, an indefinite number of trajectories exist. The fuzzy function is illustrated with four crisp parameters.

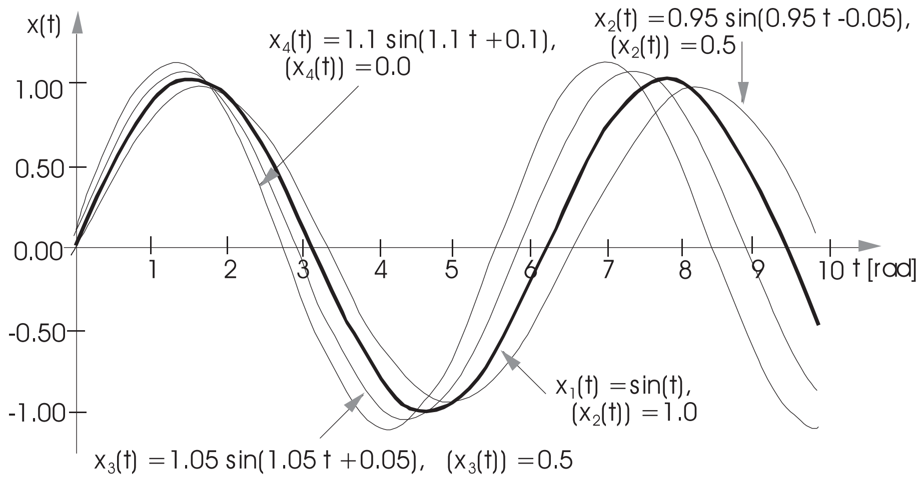


Figure 3: Some trajectories of the fuzzy function $\mathrm{x}(\underline{\tilde{\mathrm{s}}}, \mathrm{t}) = \tilde{\mathrm{a}} \cdot \sin(\tilde{\omega}_0 \mathrm{t} + \tilde{\mathrm{b}})$

3 Uncertainty Quantification

The goal of uncertainty quantification is to assign an appropriate mathematical model to real-world information with respect to objective and subjective uncertainty. The choice of an appropriate mathematical model primarily depends on the characteristics of the available information.

3.1 Fuzziness

There is no general algorithm for fuzzification, that is, for quantifying uncertainty with the model fuzziness. In order to specify a fuzzy set $\tilde{A}$ on the fundamental set $\underline{\mathbf{X}}$ it is necessary to formulate a criterion that is more or less satisfactorily fulfilled by the elements $\underline{x} \in \underline{\mathbf{X}}$. This criterion may either represent an uncertain proposition or an event. In order to specify the membership function $\mu_A(\underline{x})$, all $\underline{x} \in \underline{\mathbf{X}}$ are gradually assessed in relation to the formulated criterion. Generally, it is appropriate to choose simple functional formulations such as linear or polygonal types for the $\mu_A(\underline{x})$. In view of engineering applications, the uncertainty is usually bounded, e.g., by physical restrictions or by reasonable limits, so that the specified fuzzy sets are also bounded, which is advantageous for their numerical processing in a fuzzy analysis with α-level optimization.

Primarily, fuzzification is a subjective assessment, which depends on the available information. In this context, four types of information are distinguished to formulate at least course guidelines for fuzzification. If the information consists of various types, different fuzzification methods may be combined.

3.1.1 Information type I – sample of very small size

The membership function is specified on the basis of existing data comprising elements of a sample. The assessment criterion for the elements $\underline{x}$ is directly related to numerical values derived from $\underline{\mathbf{X}}$. An initial draft for a membership function may be generated with the aid of simple interpolation algorithms applied to the objective information, e.g., represented by a histogram. This is subsequently adapted, corrected, or modified by means of subjective aspects.

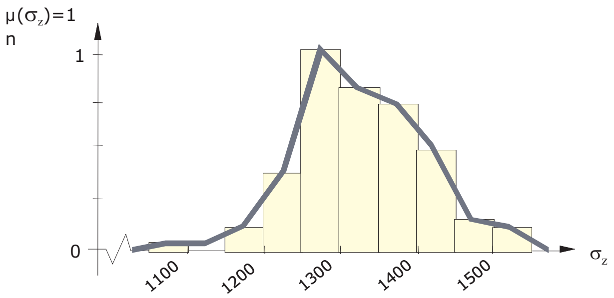


Figure 4: Fuzzification of information from a very small sample

3.1.2 Information type II – linguistic assessment

The assessment criterion for the elements $\underline{x}$ of $\underline{\mathbf{X}}$ may be expressed using linguistic variables and associated terms, such as "low" or "high". As numerical values are required for a fuzzy analysis, it is necessary to transform the linguistic variables to a numerical scale (Fig. 5). By combining the terms of a linguistic variable with modifiers, such as "very" or "reasonably", a wide spectrum is available for the purpose of assessment.

3.1.3 Information type III – single uncertain measured value

If only a single numerical value from $\underline{\mathbf{X}}$ is available as an uncertain result of measurement m, the assessment criterion for the elements $\underline{x}$ may be derived from the uncertainty of the measurement, which is quantified on the assigned numerical scale. The uncertainty of the measurement is obtained as a "gray zone" comprising more or less trustworthy values. This can be induced, e.g., by the imprecision of a measurement device or by a not clearly specifiable measuring point.

The experimenter evaluates the uncertain observation for different membership levels. For the level $\mu_A(\underline{x})$ a single measurement or a measurement interval is specified in such a way that the observation may be considered to be "as crisp as possible". For the level of the support, $\mu_A(\underline{x})$=0, a measurement

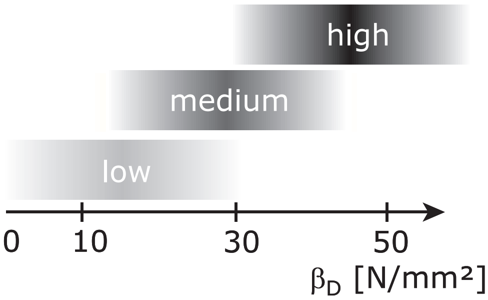
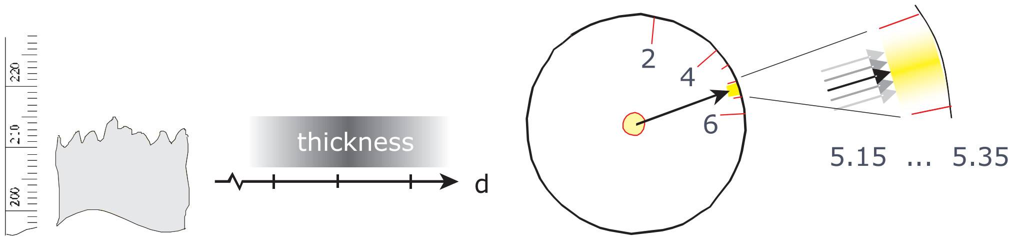


Figure 5: Fuzzification of information from a linguistic assessment

interval is determined that contains all possible measurements within the scope of the observation. An assessment of the uncertain measurements for intermediate levels is left up to the experimenter. The membership function is generated by interpolation or by connecting the determined points $(\underline{x},\ \mu_A(\underline{x}))$.



(a) imprecise measuring point (b) imprecision of the measuring device

Figure 6: Fuzzification of a single uncertain measurement

3.1.4 Information type IV – knowledge based on experience

The specification of a membership function generally requires the consideration of opinions of experts or expert groups, of experience gained from comparable problems, and of additional information where necessary. Also, possible errors in measurement, and other inaccuracies attached to the fuzzification process may be accounted for. These subjective aspects generally supplement the initial draft of a membership function. If neither reliable data nor linguistic assessments are available, fuzzification depends entirely on estimates by experts. As an example, consider a single measurement carried out under dubious conditions, which only yields some plausible value range.

In those cases, a crisp set may initially be specified as a kernel set of the fuzzy set. The boundary regions of this crisp kernel set are finally "smeared" by assigned membership values $\mu_A(\underline{x}) < 1$ to elements close to the boundary and leading the branches of $\mu_A(\underline{x})$ beyond the boundaries of the crisp kernel set monotonically to $\mu_A(\underline{x}) = 0$. By this means elements that do not belong to the crisp kernel set, but are located "in the proximity" of the latter, are also assessed with membership values of $\mu_A(\underline{x})$>0. This approach may be extended by selecting several crisp kernel sets for different membership levels (α-level sets) and by specifying the $\mu_A(\underline{x})$ in level increments.

3.2 Randomness

Data-based information which is characterized by random fluctuations may be described with the aid of a traditional probabilistic model. This is associated with certain requirements, e.g., regarding environmental conditions or compliance with the i.i.d. paradigm. If all the conditions are met, the entire variety of mathematical statistics with its established methods from estimation and test theory can be employed to specify and to quantify the uncertainty. The desire for considering well-defined subjective assessments in terms of probability requires an extended probabilistic model. As a basis the model of subjective probability, in particular, in the form of a Bayesian approach is frequently proposed for this purpose. Probabilistic models are comprehensively described in a large quantity of books, e. g., [19, 14, 7, 12, 28, 18, 11, 27, 32, 13, 29, 33].

4 Fuzzy analysis

All fuzzified structural parameters are together introduced into fuzzy structural analysis. Fuzzy structural analysis implies the analysis of a structure with the aid of a crisp (or uncertain) algorithm applied to fuzzy values for input and model parameters. In fuzzy structural analysis the deterministic algorithms for static and dynamic computations are adopted as a deterministic fundamental solution. The fuzziness of uncertain input and model parameters is processed on the basis of the developed α-level optimization [23]. The solution technique is formulated in terms of a modified evolution strategy that targets at a minimal computational effort. This concept permits to implement an arbitrary nonlinear deterministic fundamental solution without any special properties.

The developed algorithms of fuzzy structural analysis yield sets of discrete points in the space of the fuzzy result values. These points are evaluated by membership values. Additionally, the parameter coordinates in the space of the fuzzy input values and fuzzy model parameters which belong to these result points are known.

4.1 α-level discretization of fuzzy values

As a basis for α-level optimization the concept of α-discretization is applied to represent fuzzy sets numerically with the aid of their α-level sets. The mapping of onto is then formulated as an optimization problem and solved with a modified evolution strategy. A post-computation is performed to improve the performance of the procedure. This combination represents a numerically efficient tool for fuzzy analysis. The numerical cost approximately linearly increases with the number of fuzzy input variables.

An α-level set $\underline{A}_{\alpha_k}$ of the fuzzy set $\tilde{A}$ is defined as a crisp set associated

$$\underline{A}_{\alpha_k} = \{\underline{x} \in \underline{X} \mid \mu_A(\underline{x}) \geq \alpha_k\} \tag{9}$$

with a selected real number $\alpha_k \in (0,1]$. All α-level sets are crisp subsets of the support $S(\tilde{A})$. For several α-level sets of the same fuzzy set the following inclusion property holds

$$\underline{A}_{\alpha_k} \subseteq \underline{A}_{\alpha_1} \forall \quad \alpha_i; \alpha_k(0,\ 1], \quad \alpha_i \leq \alpha_k. \tag{10}$$

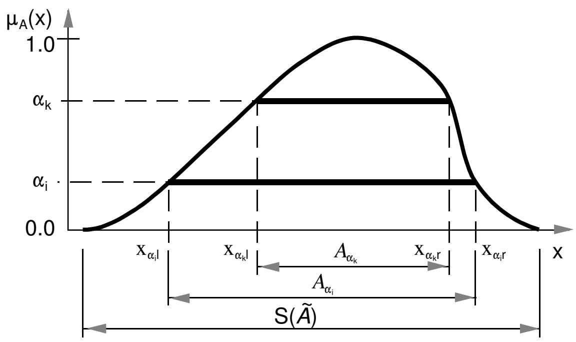


Figure 7: α-level sets

If – in the one-dimensional case – the fuzzy set $\tilde{A}$ is convex, each α-level set $\tilde{A}_k$ is a connected interval $[x_{\alpha_k,l},\ x_{\alpha_k,r}]$ with

$$x_{\alpha_k,l} = min[x \in X \mid \mu_A(x) \geq \alpha_k], \quad x_{\alpha_k,r} = max[x \in X \mid \mu_A(x) \geq \alpha_k] \tag{11}$$

The concept of α-discretization provides a numerically efficient representation of fuzzy sets. In contrast to the extension principle, the elements of a fuzzy set are not considered separately one after the other, which means a discretization of the support, but the membership scale is now discretized, and the associated α-level sets are considered. For a sufficiently high number of α-levels a fuzzy set can be completely represented as a set of its α-level sets, that is, by its *α-discretization*

$$\underline{\tilde{A}} = \left\{ (\underline{A}_{\alpha_k}, \mu(\underline{A}_{\alpha_k}) \mid \mu(\underline{A}_{\alpha_k}) = \alpha_k \forall \alpha_k \in (0,\ 1]) \right\} \tag{12}$$

For each α-level, the associated α-level sets A_{i,α_k} of the fuzzy input variables $\tilde{x}_i = \tilde{A}_i$ constitute an n-dimensional crisp subspace $\underline{X}_{\alpha_k}$ of the x-space, see Fig. 8.

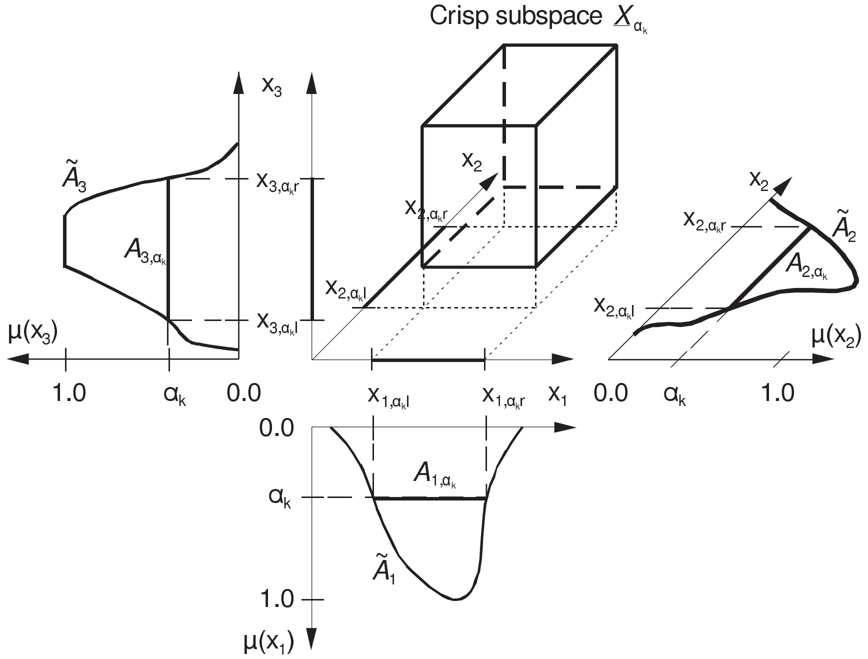


Figure 8: Constitution of crisp subspaces $\underline{X}_\alpha$

4.2 α-level optimization

All fuzzy input variables are discretized using the same number of α-levels α_k, k = 1, ..., r to form the associated crisp subspaces. With the aid of the mapping model

$$M: \quad \underline{z} = f(x_1, \ldots, x_n) \tag{13}$$

crisp input vectors $\underline{x}$ are transformed into the result space. The elements z_j of the associated crisp result vectors $z = (z_1, ..., z_m) = f(x)$ represent elements of the α-level sets $B_{j,\alpha_k}$of the fuzzy result variables on the α-level α_k. The mapping of all elements of yields the crisp subspace $\underline{Z}_{\alpha_k}$ of the z-space. Once the largest element $z_{j,\alpha_k r}$ and the smallest element $z_{j,\alpha_k l}$ of the α-level set have been found, two points of the membership function $\mu(z_j) = \mu_{B_j}(z_j)$ are known, see Fig. 9. These points are sufficient to completely describe convex fuzzy result variables.

The search for the smallest and largest result elements on each α-level represents an optimization problem and is thus referred to as *α-level optimization*. The objective functions

$$z_j = f_j(x_1;\ \ldots;\ x_n) \Rightarrow Max \mid (x_1;\ \ldots;\ x_n) \in \underline{X}_\alpha \tag{14}$$

$$z_j = f_j(x_1;\ \ldots;\ x_n) \Rightarrow Min \mid (x_1;\ \ldots;\ x_n) \in \underline{X}_\alpha \tag{15}$$

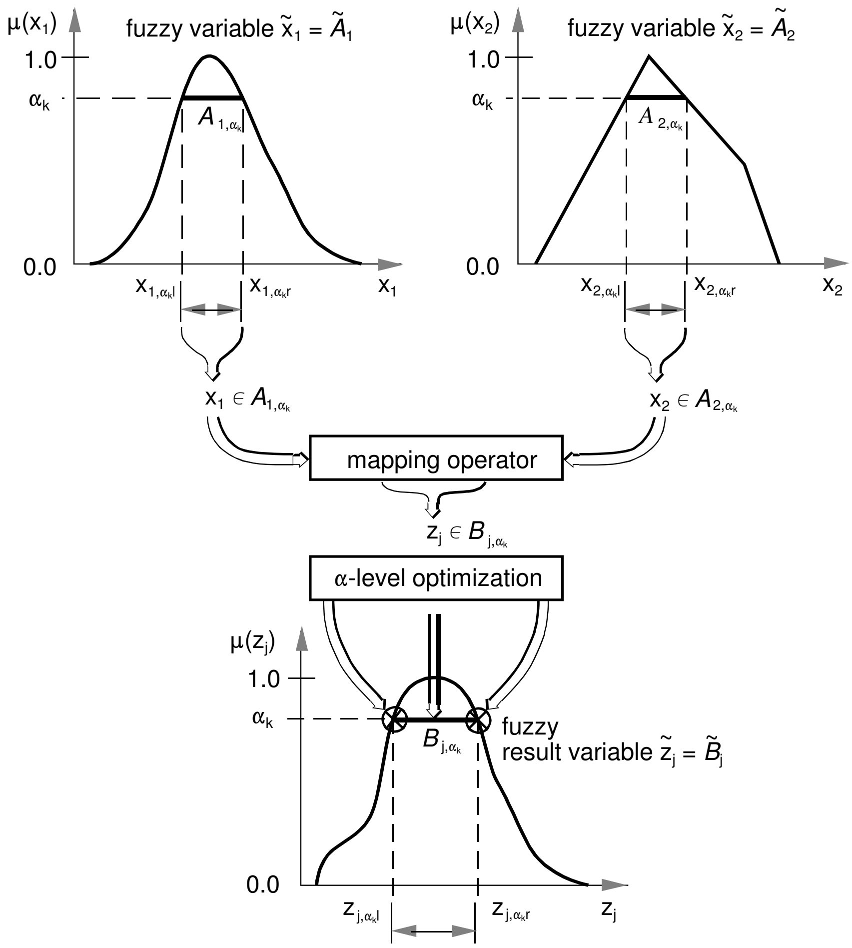


Figure 9: α-level optimization

must be satisfied by the optimum points $\underline{x}_{opt} \in \underline{X}_{\alpha_k}$. As no requirements are formulated for the mapping model, the optimization problem has no special but very general properties. That is, the optimum $\underline{x}_{opt}$ points may be located arbitrarily in the input subspaces . To solve this general optimization problem a modified evolution strategy has been developed. If every crisp subspace is a connected set, and if the mapping model is continuous and single-valued, α-level optimization yields exact results as the fuzzy result variables $\tilde{z}_j$ are then convex fuzzy sets. If the mapping model is not continuous or single-valued, α-level optimization yields exact envelope curves of the actual membership functions of the fuzzy result variables, which is commonly sufficient in engineering.

4.2.1 Modified evolution strategy

The modified evolution strategy is a numerical evolution-based optimization method that is particularly suitable for solving α-level optimization within the scope of a general fuzzy analysis. It does not require any special properties of the objective function and is low-sensitive to noise. The numerical procedure possesses a simple structure and can be applied very flexibly in dependence on the problem by adjusting several effective control parameters. This concept permits an implementation of arbitrary nonlinear algorithms as mapping models, e.g. for structural analyzes, into a fuzzy analysis with α-level optimization.

The primary algorithm of the modified evolution strategy is formulated for continuous coordinates and constant constraints, which corresponds to a fuzzy analysis with non-interactive fuzzy input variables. An extension to more general conditions, in particular, for dealing with discrete optimization problems is straightforward.

The solution technique targets at a minimum computational cost. This increases approximately linearly with the number of dimensions of the problem. The modified evolution strategy may be characterized as a generally applicable, numerically efficient and robust optimization technique.

Numerical procedure. The optimization problem is described by the decision/design variables x_i, the objective function $z = f(x_1,...,x_n)$, and some constraints specifying the permissible domain for the x_i. With respect to a fuzzy analysis with α-level optimization, the decision variables are defined by the input variables $\tilde{x}_i,\ i = 1,\ldots,\ n$, the objective function is described by the mapping model M, and the constraints are given by $(x_1,...,x_n) \in \underline{X}_{\alpha_k}$, see Fig. 10.

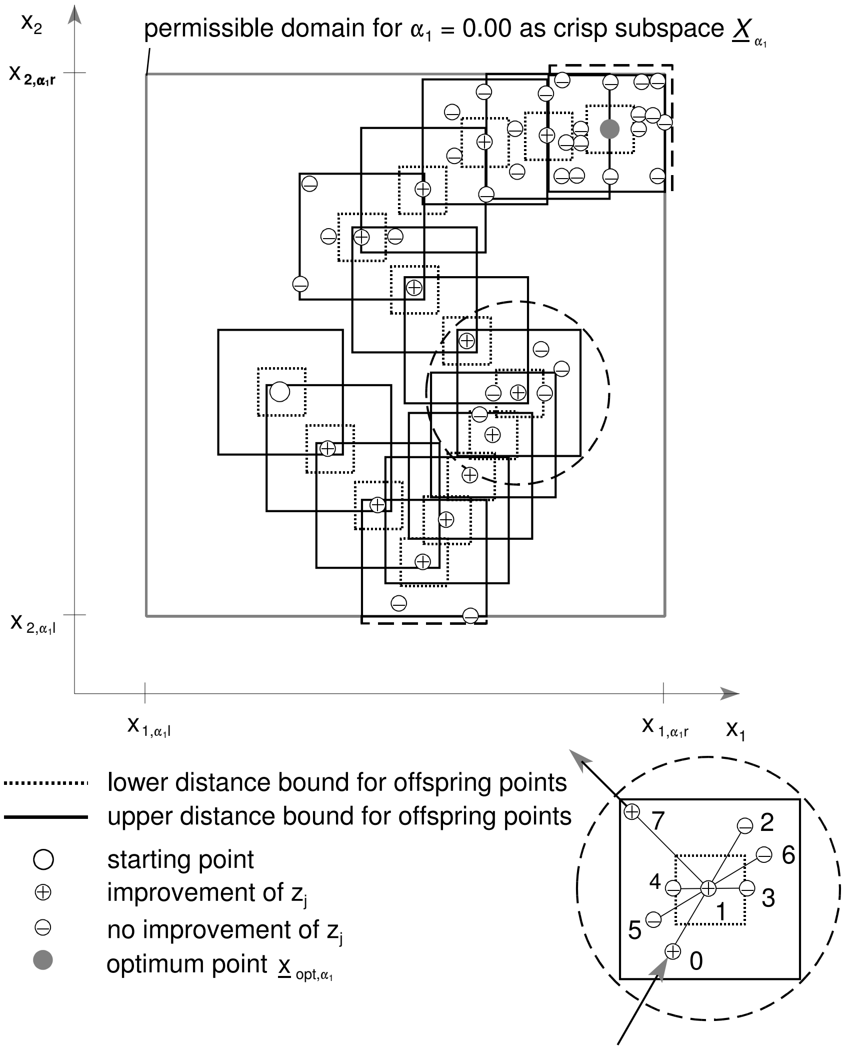


Figure 10: Modified evolution strategy, non-interactive decision/input variables

The numerical procedure is formulated as a (1+1) evolution strategy that is modified by incorporating some elements from a gradient descend method and from a Monte Carlo method.

The starting point representing the first parent point $\underline{x}^{[q]} = \underline{x}^{[0]}$ is specified with the aid of uniform distribution over the permissible domain. For generating offspring points $\underline{x}^{[q+1]}$ a maximum and a minimum distance from the parent point $\underline{x}^{[q]}$ are specified for each coordinate x_i, which forms a local search domain. A definition of max_d_i and min_d_i by compressing transfers its proportions to the local search domain.

A first offspring point $\underline{x}^{[q+1]}$ of $\underline{x}^{[q]}$ is generated within this local search domain by means of a uniform distribution. Its objective function value $z^{[q+1]} = f(\underline{x}_1^{[q+1]},...,\underline{x}_n^{[q+1]})$ is checked for an improvement compared with $z^{[q]} = f(\underline{x}_1^{[q]},...,\underline{x}_n^{[q]})$ of the parent point. If an improvement is obtained, the search proceeds along the randomly selected direction until no further improvement in the objective function value occurs. If the boundary of the permissible domain is crossed over, the non-permissible points are drawn back on the permissible boundary to comply with the constraints $(x_1,...,x_n) \in \underline{X}_{\alpha_k}$.

If the randomly determined offspring point $\underline{\mathrm{x}}^{[q+1]}$ does not lead to an improvement, the next offspring point $\underline{\mathrm{x}}^{[q+2]}$ is positioned at the same distance as $\underline{\mathrm{x}}^{[q+1]}$ from the parent point $\underline{\mathrm{x}}^{[q]}$ but in the opposite direction

$$\underline{\mathrm{x}}^{[q+2]} = 2 \cdot \underline{\mathrm{x}}^{[q]} - \underline{\mathrm{x}}^{[q+1]} \quad \left| \begin{array}{ll} z^{[q+2]} \geq z^{[q]} & \text{for minimum search} \\ z^{[q+2]} \leq z^{[q]} & \text{for maximum search} \end{array} \right. \tag{16}$$

If the offspring point $\underline{\mathrm{x}}^{[q+2]}$ also shows no further improvement in the value of $\underline{\mathrm{z}}^{[q+1]}$, the next point $\underline{\mathrm{x}}^{[q+3]}$ is again determined randomly starting from $\underline{\mathrm{x}}^{[q]}$ as in the computation of $\underline{\mathrm{x}}^{[q+1]}$, and so on.

This procedure is continued until no further improvement is achieved within a given number of tests. Then, the distance bounds are reduced by a factor to refine the search. After some refinements and further unsuccessful steps the procedure is terminated. The last parent point is taken as the optimum point $\underline{x}_{opt}$.

To increase the numerical efficiency, existing points $\underline{x}^{[q]}$ with known objective function values $z^{[q]}$ from previous optimizations are reused. In a defined close neighborhood of new points this available information is exploited instead of evaluating the objective function.

A further efficiency feature is caused by the determination procedure for offspring points. This guides the search, by higher probability, towards the corners of the permissible domain , where the $\underline{x}_{opt}$ are located in the case of monotonic mappings. Also, the direction of the largest extent of is preferred compared with other directions.

Finally, several control parameters are defined to adjust the optimization procedure to the problem in each particular case and thus to optimize the numerical search performance. These control parameters specify, for example, the size of the local search domain, the neighborhood for reusing existing points, the refinement factor for the local search domain, and termination criteria.

5 Robustness design

5.1 Robustness investigations

Robustness and sensitivity analysis of structural designs with respect to the uncertainty of design parameters. In order to discriminate between robustness and sensitivity of a structural design a clear definitions of each other is required.

Robustness. A structural design is considered as being robust if deviations from the desired values of structural parameters yields to reasonable low deviations of structural responses. As robustness can not be measured in absolute terms only relative measure can be formulated.

Sensitivity. The sensitivity describes the amount of influence of structural design parameters regarding structural results. As robustness the sensitivity of a structural design can only be measured relatively.

In most applications there is no knowledge about the exact value of the deviations of the structural parameters. Thus they are interpreted as uncertain and can be described with different uncertainty models.

If the uncertainty model fuzziness is applied, fuzzy design parameters are mapped to fuzzy structural results with the aid of the fuzzy structural analysis. The relation between the uncertainty of design and result parameters can be assessed with an analog to Shannon´s entropy and is defined by

$$\mathrm{H_u}(\tilde{A}_\mathrm{i}) = -\mathrm{k} \cdot \int_{\mathrm{x}=-\infty}^{\mathrm{x}=+\infty} [\mu(\mathrm{x}) \cdot \ln(\mu(\mathrm{x})) + (1-\mu(\mathrm{x})) \cdot \ln(1-\mu(\mathrm{x}))] \, \mathrm{dx} \tag{17}$$

The modified Shannon's entropy represents the "steepness" of the membership function. When assessing a crisp set the measure value $H_u(\tilde{A}_i) = 0$ is obtained. The most uncertain set where all elements of this set have a membership value of $\mu(x) = 0.5$ (except the mean value) the measure value $H_u(\tilde{A}_i)$ yields its maximum value.

A relative sensitivity measure is defined by ratio of the Shannon's entropy of the fuzzy design parameters $\underline{\tilde{x}}$ to the Shannon's entropy of the fuzzy result $\tilde{z}_j$

$$B_j^s = \sum_k^n \frac{H_u(\tilde{x}_k)}{H_u(\tilde{z}_j)} \tag{18}$$

This sensitivity measure can be used to evaluate different structural design variants.

5.2 Fuzzy Cluster Design

Fuzzy cluster design permits the design of a structure on the basis of fuzzy analysis. Fuzzy analysis provides a point to point mapping between points in the input space and points in the result space. If points in the result space fulfill design constraints then the related point in the input space is a potential design parameter. As the known point-to-point mapping is discrete points are merged into cluster, see Fig. 11.

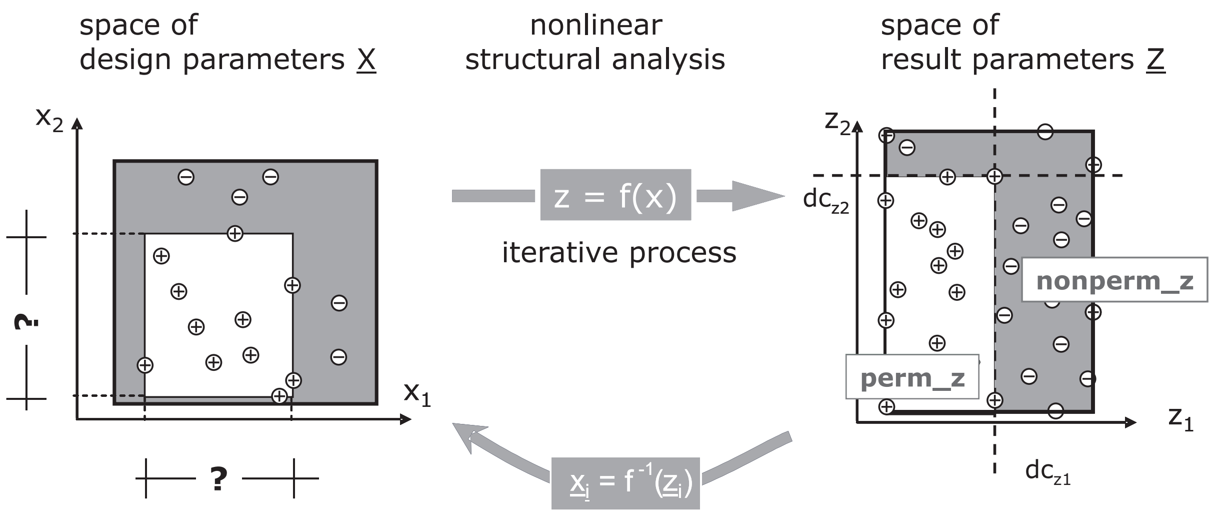


Figure 11: Problem to be solved: Determination of convex sets of permissible design vectors

Starting from this point the algorithmic procedure of Fuzzy Cluster Design can be described with the following three steps:

1. Convex and connected sets of permissible and non-permissible points which fulfill have to be determined. For that purpose crisp cluster algorithms and fuzzy cluster algorithms offer a suitable basis. These detected permissible clusters represent possible alternative design variants.
2. These possible alternative design variants are modeled as fuzzy input values and are again introduced into fuzzy structural analysis. The obtained associated fuzzy results are compared to given design constraints again.
3. After the determination of different alternative design variants that all meets the given design constraints the alternative design variants are assessed together with their results in consideration of robustness and distance to design constraints.

An introduction to fuzzy cluster design and its applications can be found in [21, 4, 3].

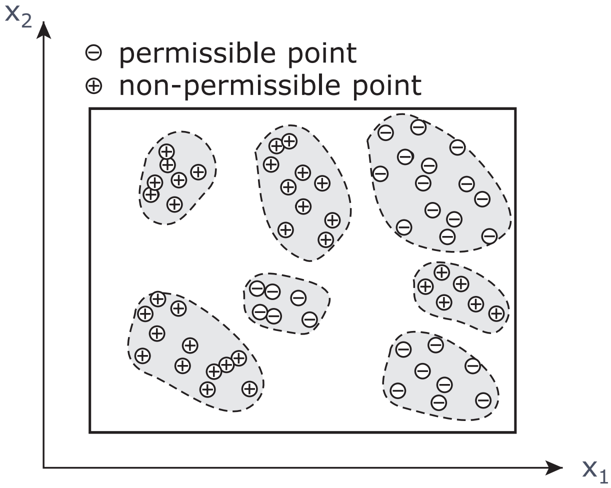


Figure 12: Detected permissible and non-permissible clusters in the design space

6 Conclusions

In this paper the uncertainty model fuzziness and suggestions for uncertainty quantification of real-world data has been provided. The uncertainty model fuzziness lends itself to describing imprecise, subjective, linguistic, and expert-specified information. It is capable of representing dubious, incomplete, and fragmentary information and can additionally incorporate objective, fluctuating, and data-based information in the fuzziness description. Requirements regarding special properties of the information do generally not exist. That is, for example, environmental conditions can be unknown or arbitrarily fluctuating. With respect to the regularity of information within the uncertainty, the uncertainty model fuzziness is less rigorous in comparison with probabilistic models. It specifies a lower information content and thus possesses the advantage of requiring less information for an adequate uncertainty quantification.

Further the numerical algorithm fuzzy analysis that maps fuzzy structural parameters onto fuzzy results with an arbitrary (nonlinear) computational model has been presented. The major benefit of the fuzzy analysis in contrast to the stochastic analysis is that worst case / best case studies, robustness investigations and uncertain structural analysis with parameters whose uncertainty has been identified as fuzziness can be performed numerical efficiently. Exceptional structural responses are taken into account and determined with the aid of the α-level optimization. The probability to detected such exceptional structural responses with a stochastic analysis is equal to zero, but for prediction the robustness of a structure they are fundamental.

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